Section 2.3 The Chain Rule (Minimum problems: 17 – 38 odds)

This section covers a new derivative rule called THE CHAIN RULE.

The theory behind the rule is harder than applying the rule in reality.

I will do a short introduction to the theory. We will spend most of the section applying the rule.

Example – Composition, Algebra review.

Complete the table by finding $f[g(x)]$, and do not simplify your answer!!!

	$f(x)$	$g(x)$	$f[g(x)]$
Example 1	$f(x) = x^4$	$g(x) = 2x + 3$	
Example 2	$f(x) = 5x^2$	$g(x) = x + 5$	
Example 3	$f(x) = 9\sqrt{x}$	$g(x) = x^2 + 5x - 7$	

To complete the table replace the "x" in the f-function with the right side of the g-function.

Completed table:

	$f(x)$	$g(x)$	$f[g(x)]$
Example 1	$f(x) = x^4$	$g(x) = 2x + 3$	$f[g(x)] = (2x + 3)^4$
Example 2	$f(x) = 5x^2$	$g(x) = x + 5$	$f[g(x)] = 5(x + 5)^2$
Example 3	$f(x) = 9\sqrt{x}$	$g(x) = x^2 + 5x - 7$	$f[g(x)] = 9\sqrt{x^2 + 5x - 7}$

We now need to go backwards.

Example – Composition in reverse.

Complete the table by creating two functions $f(x) and\ g(x)$ whose composition is the given function $f[g(x)]$

	$f(x)$	$g(x)$	$f[g(x)]$
Example 1			$f[g(x)] = (x+2)^5$
Example 2			$f[g(x)] = 6(3x-4)^3$
Example 3			$f[g(x)] = 9\sqrt{7x+5}$

To complete the table make the

$g - function$ the inside of the parenthesis, or expression under the radical.

$f - funtion$ the problem with the inside of the parenthesis or the expression under the radical changed to x.

	$f(x)$	$g(x)$	$f[g(x)]$
Example 1	$f(x) = x^5$	$g(x) = x+2$	$f[g(x)] = (x+2)^5$
Example 2	$f(x) = 6x^3$	$g(x) = 3x-4$	$f[g(x)] = 6(3x-4)^3$
Example 3	$f(x) = 9\sqrt{x}$	$g(x) = 7x+5$	$f[g(x)] = 9\sqrt{7x+5}$

This is the formal definition of THE CHAIN RULE:

Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Example: Use the chain rule to find the derivative of $h(x) =$
$(7x + 5)^4$

We need to consider the given function as the composition of the functions $f \ and \ g$

$h(x) = (7x + 5)^4$

$f[g(x)] = (7x + 5)^4$

First create the functions $f(x) \ and \ g(x)$

$f(x) = x^4$

$g(x) = 7x + 5$

Next, we find $f'(x)$

$f'(x) = 4x^3$

Now we find
$f'(g(x))$ *by replacing the x in the derivative with the right side of the g − function*

$f'(g(x)) = 4(7x + 5)^3$

Lastly, we find $g'(x)$

$g'(x) = 7$

To find the derivative we multiply the last two results.

$h'(x) = 4(7x + 5)^3 * 7$

Answer: $h'(x) = 28(7x + 5)^3$

The last example is the proper way to find a derivative using the chain rule. Realistically, we can find a way to use the chain rule to find derivatives with problems that have parenthesis with exponents quickly.

CHAIN RULE short cut to find a derivative of a problem written in the form:

$f(x) = a[g(x)]^n$ (a is a constant, a number without a letter)

$$f(x) = a(inside\ parenthesis)^n$$

$$f'(x) = n * a * g'(x)[g(x)]^{n-1}$$

$$f'(x) = n * a * (derivative\ of\ inside\ of\ parenthesis)(original\ parenthesis)^{n-1}$$

Example:

Find the derivative of $f(x) = 5(3x + 8)^4$

$a = 5$
$g(x) = 3x + 8$
$g'(x) = 3$

$f'(x) = 4 * 5 * 3(3x + 8)^3$

Answer: $f'(x) = 60(3x + 8)^3$

Example:

Find the derivative of $f(x) = (3x^2 - 7x + 5)^2$

$a = 1$
$g(x) = 3x^2 - 7x + 5$
$g'(x) = 6x - 7$

Answer: $f'(x) = 2(6x - 7)(3x^2 - 7x + 5)$

Now the hard part. Some problems require us to use both the product rule and the chain rule to compute a derivative.

Example: Find $\frac{dy}{dx}$ given: $y = 8x(2x+1)^3$

We need to first set up the product rule.

First factor $8x$	Second Factor $(2x+1)^3$
Derivative 8	Derivative $a = 1$ $g(x) = 2x + 1$ $g'(x) = 2$ $3 * 2(2x+1)^2$ $6(2x+1)^2$
Cross multiply down $8x * 6(2x+1)^2$ $48x(2x+1)^2$	Cross multiply up $8(2x+1)^3$

$y' = 8(2x + 1)^3 + 48x(2x + 1)^2$

Let us pull out the common factor one step at a time, although I would do this in one step.

First pull out the common factor of 8

$y' = 8[(2x + 1)^3 + 6x(2x + 1)^2]$

Now pull out the trickier common factor of $(2x + 1)^2$

$y' = 8(2x + 1)^2[(2x + 1)^1 + 6x]$

$y' = 8(2x + 1)^2(2x + 1 + 6x)$

Answer: $y' = 8(2x + 1)^2(8x + 1)$

Last example: $f(x) = 5(3x + 8)^4$

a) Find all values of x where the tangent line is horizontal

b) Find the equation of the tangent line to the graph of the function for the values of x found in part a.

a) Find all values of x where the tangent line is horizontal

Solve: $f'(x) = 0$

We found the necessary derivative in a previous example:

$f'(x) = 60(3x + 8)^3$

Now solve:

$60(3x + 8)^3 = 0$

$60 = 0 \;\; (no\ solution)$

$(3x + 8)^3 = 0$

Sufficient to solve

$3x + 8 = 0$

$3x = -8$

$x = -\frac{8}{3}$

Part “a” answer: $x = -\frac{8}{3}$

b) Find the equation of the tangent line to the graph of the function for the values of x found in part a.

So far we know $m = 0, since\ line\ is\ horizontal \quad and\ x = -\frac{8}{3}$

We need to find the y-coordinate of the point.

$y = f\left(-\frac{8}{3}\right) = 5\left(3 * \frac{-8}{3} + 8\right)^4 = 0$ (simplified on my calculator)

$point\ \left(-\frac{8}{3}, 0\right)\ slope\ m = 0$

$y - 0 = 0(x - \left(-\frac{8}{3}\right))$

$y = 0\left(x + \frac{8}{3}\right)$

Answer: $y = 0$

EQUATIONS OF HORIZONTAL TANGENT LINES WILL
ALWAYS BE OF THE FORM $y = y - coordinate\ of\ the\ point$

#1-10: Find $f[g(x)]$, and do not simplify your answer!!!

1) $f(x) = x^3$; $g(x) = x^2 + 1$

2) $f(x) = x^4$; $g(x) = 4x^2 + 5$

$answer$: $f[g(x)] = (4x^2 + 5)^4$

3) $f(x) = 5x^2; \quad g(x) = 3x - 4$

4) $f(x) = 4x^2; \quad g(x) = 2x - 5$

$answer: f[g(x)] = 4(2x - 5)^2$

#1-10: Find $f[g(x)]$, and do not simplify your answer!!!

5) $f(x) = 7x^{2/3}$; $g(x) = 5x + 4$

6) $f(x) = 8x^{3/4}$; $g(x) = 6x + 4$

$answer: f[g(x)] = 8(6x + 4)^{\frac{3}{4}}$

7) $f(x) = e^x; \quad g(x) = x^2 + 2x + 1$

8) $f(x) = e^x; \quad g(x) = 4x^2 + x - 5$

$answer: f[g(x)] = e^{4x^2+x-5}$

#1-10: Find $f[g(x)]$, and do not simplify your answer!!!

9) $f(x) = ln(x)$; $g(x) = 3x + 5$

10) $f(x) = ln(x)$; $g(x) = 2x - 7$

$answer$: $f[g(x)] = \ln{(2x - 7)}$

#11-16: Create two functions $f(x)$ and $g(x)$ whose composition is the given function $f[g(x)]$

11) $f[g(x)] = (7x-3)^2$

12) $f[g(x)] = (8x-5)^2$

$answer$ $f(x) = x^2$ $g(x) = 8x - 5$

13) $f[g(x)] = 2(4x+7)^5$

14) $f[g(x)] = 3(5x+4)^4$

$answer$: $f(x) = 3x^4 \quad g(x) = 5x+4$

#11-16: Create two functions $f(x)$ and $g(x)$ whose composition is the given function $f[g(x)]$

15) $f[g(x)] = \sqrt{x+5}$

16) $f[g(x)] = \sqrt[3]{7x+1}$

$answer$ $f(x) = \sqrt[3]{x}$ $g(x) = 7x + 1$

#17-24: Use the Chain rule to find the derivative of each function.

$$\frac{d}{dx} f[g(x)] = g'(x) * f'[g(x)]$$

CHAIN RULE short cut to find a derivative of a problem written in the form:

$$f(x) = a[g(x)]^n$$

$$f(x) = a(inside\ parenthesis)^n$$

$$f'(x) = n * a * g'(x)[g(x)]^{n-1}$$

$$f'(x) = n * a * (derivative\ of\ inside\ of\ parenthesis)(original\ parenthesis)^{n-1}$$

17) $h(x) = (7x - 3)^2$

18) $h(x) = (8x - 5)^2$

$answer\ \ h'(x) = 16(8x - 5)$

CHAIN RULE short cut to find a derivative of a problem written in the form:

$$f(x) = a[g(x)]^n$$

$$f(x) = a(inside\ parenthesis)^n$$

$$f'(x) = n * a * g'(x)[g(x)]^{n-1}$$

$$f'(x) = n * a * (derivative\ of\ inside\ of\ parenthesis)(original\ parenthesis)^{n-1}$$

19) $h(x) = 2(4x + 7)^5$

20) $h(x) = 3(5x + 4)^4$

answer $h'(x) = 60(5x + 4)^3$

CHAIN RULE short cut to find a derivative of a problem written in the form:

$$f(x) = a[g(x)]^n$$

$$f(x) = a(inside\ parenthesis)^n$$

$$f'(x) = n * a * g'(x)[g(x)]^{n-1}$$

$$f'(x) = n * a * (derivative\ of\ inside\ of\ parenthesis)(original\ parenthesis)^{n-1}$$

21) $h(x) = 4(2x - 1)^3$

22) $h(x) = 2(5x - 6)^3$

Answer: $h'(x) = 30(5x - 6)^2$

CHAIN RULE short cut to find a derivative of a problem written in the form:

$$f(x) = a[g(x)]^n$$

$$f(x) = a(inside\ parenthesis)^n$$

$$f'(x) = n * a * g'(x)[g(x)]^{n-1}$$

$$f'(x) = n * a * (derivative\ of\ inside\ of\ parenthesis)(original\ parenthesis)^{n-1}$$

23) $h(x) = (x^2 + 6x + 1)^3$

24) $h(x) = (3x^2 - 5x + 2)^3$

Answer: $h'(x) = 3(6x - 5)(3x^2 - 5x + 2)^2$

CHAIN RULE short cut to find a derivative of a problem written in the form:

$$f(x) = a[g(x)]^n$$

$$f(x) = a(inside\ parenthesis)^n$$

$$f'(x) = n * a * g'(x)[g(x)]^{n-1}$$

$$f'(x) = n * a * (derivative\ of\ inside\ of\ parenthesis)(original\ parenthesis)^{n-1}$$

#25-46: Find the derivative of each function.

25) $y = 5x(2x-4)^3$

26) $y = 5x(7x+1)^3$

We need to first set up the product rule. We will need the chain rule for one of the derivatives during the product rule.

First factor	Second Factor
Derivative	Derivative
Cross multiply down	Cross multiply up

Answer: $y' = 5(7x+1)^2(56x+5)$

27) $g(t) = 6t^2(2t+5)^2$

28) $g(t) = 5t^2(4t-1)^2$

We need to first set up the product rule. We will need the chain rule for one of the derivatives during the product rule.

First factor	Second Factor
Derivative	Derivative
Cross multiply down	Cross multiply up

Answer: $g'(t) = 10t(4t-1)(8t-1)$

29) $h(y) = (6y - 3)(5y + 4)^2$

30) $f(y) = (2y - 3)(3y - 4)^2$

We need to first set up the product rule. We will need the chain rule for one of the derivatives during the product rule.

First factor	Second Factor
Derivative	Derivative
Cross multiply down	Cross multiply up

Answer: $f'(y) = 2(3y - 4)(9y - 13)$

31) $y = \frac{2}{(3x-4)^2}$

32) $y = \frac{5}{(2x-9)^3}$

This needs the quotient rule. We will need the Chain Rule for one of the derivatives during the quotient rule.

Denominator	Numerator
Derivative	Derivative
$cross\ multiply\ top\ down$	$cross\ multiply\ bottom\ up$

Create a fraction. Place the expressions in the numerator with a subtraction between. Place the square of the denominator in the denominator

Answer: $y' = \frac{-30}{(2x-9)^4}$

33) $y = \frac{2x}{(3x-4)^4}$

34) $y = \frac{5x}{(2x-9)^3}$

This needs the quotient rule. We will need the Chain Rule for one of the derivatives during the quotient rule.

Denominator	Numerator
Derivative	Derivative
cross multiply top down	*cross multiply bottom up*

Create a fraction. Place the expressions in the numerator with a subtraction between. Place the square of the denominator in the denominator

Answer: $y' = \frac{-20x-45}{(2x-9)^4}$

#35-40:

a) Find all values of x where the tangent line is horizontal

b) Find the equation of the tangent line to the graph of the function for the values of x found in part a.

EQUATIONS OF HORIZONTAL TANGENT LINES WILL ALWAYS BE OF THE FORM $y = y - coordinate\ of\ the\ point$

35) $f(x) = (2x-3)^2$

36) $f(x) = (3x-4)^2$

Answer: a) $x = 4/3$ b) $y = 0$

#35-40:

a) Find all values of x where the tangent line is horizontal

b) Find the equation of the tangent line to the graph of the function for the values of x found in part a.

EQUATIONS OF HORIZONTAL TANGENT LINES WILL ALWAYS BE OF THE FORM $y = y - coordinate\ of\ the\ point$

37) $y = 5(x + 3)^4$

38) $y = 7(5x - 6)^2$

Answer: a) $x = 6/5$ b) $y = 0$